Appendix

Proof of Theorem 1

First, we prove the order of singular values is preserved in a neighborhood of the rank-r matrix M. Using Weyl's theorem, we have

$$|\sigma_i(M+\Delta)-\sigma_i| \le \|\Delta\|_F, \qquad \text{for } 1 \le i \le n.$$

For any i such that $\sigma_i > \sigma_{i+1}$: since $\|\Delta\|_F < \frac{\epsilon}{2} \le \frac{\sigma_i - \sigma_{i+1}}{2}$, the following inequality holds

$$\sigma_{i+1}(M+\Delta) < \sigma_{i+1} + \frac{\sigma_i - \sigma_{i+1}}{2} = \sigma_i - \frac{\sigma_i - \sigma_{i+1}}{2} < \sigma_i(M+\Delta).$$

Thus, the order of singular values is preserved. Moreover, since $\sigma_r(M+\Delta) - \sigma_{r+1}(M+\Delta) > 0$, the top r singular value components are unique and consequently $\mathcal{P}_r(M+\Delta)$ is unique.

Let $M = \sum_{i=1}^r \sigma_i u_i v_i^T$ be the rank-r matrix of interest. From matrix perturbation theory [1], we can describe the decomposition of the perturbed matrix

$$M+\Delta = \sum_{i=1}^r (\sigma_i+\delta_i)(u_i+\delta u_i)(v_i+\delta v_i)^T + \sum_{i=r+1}^n \delta_i(u_i+\delta u_i)(v_i+\delta v_i)^T \tag{1}$$

where $\delta_i, \delta u_i$, and δv_i have norms **in the order of** $O(\|\Delta\|_F)$. Since the top-r singular values of M are preserved under perturbation, we have $\mathcal{P}_r(M+\Delta) = \sum_{i=1}^r (\sigma_i+\delta_i)(u_i+\delta u_i)(v_i+\delta v_i)^T$ and (1) can be reorganized as

$$\mathcal{P}_r(M+\Delta) - M = \Delta - \sum_{i=r+1}^n \delta_i(u_i+\delta u_i)(v_i+\delta v_i)^T = \Delta - \sum_{i=r+1}^n u_i\delta_i v_i^T + O(\|\Delta\|_F^2). \tag{2}$$

Further, substituting $M = \sum_{i=1}^r \sigma_i u_i v_i^T$ into (1) yields

$$\Delta = \sum_{i=1}^n \left(\delta_i u_i v_i^T + \sigma_i \delta u_i v_i^T + \sigma_i u_i {\delta v_i}^T\right) + O(\|\Delta\|_F^2).$$

Then using the orthogonality of u_i, v_i, we can obtain

$$u_i^T \Delta v_i = \delta_i + \sigma_i(u_i^T \delta u_i + {\delta v_i}^T v_i) + O(\|\Delta\|_F^2), \tag{3}$$

$$u_i^T \Delta v_j = O(\|\Delta\|_F^2). \tag{4}$$

The second term on the RHS can be computed as follows

$$I = \sum_{i=1}^n (u_i+\delta u_i)(u_i+\delta u_i)^T \qquad \Rightarrow 1 = u_i^T u_i = 1 + u_i^T \delta u_i + {\delta u_i}^T u_i + O(\|\Delta\|_F^2) \qquad \Rightarrow u_i^T \delta u_i = O(\|\Delta\|_F^2)$$

Similarly, we also have $v_i^T \delta v_i = O(\|\Delta\|_F^2)$. Substituting back to (3), we get $\delta_i = u_i^T \Delta v_i + O(\|\Delta\|_F^2)$. Thus, (2) can be rewritten as

$$\mathcal{P}_r(M+\Delta) - M = \Delta - \sum_{i=r+1}^n u_i u_i^T \Delta v_i v_i^T + O(\|\Delta\|_F^2) = \Delta - U_2 U_2^T \Delta V_2 V_2^T + O(\|\Delta\|_F^2).$$

where the last equation stems from (4).

Proof of Theorem 3

Vectorizing Theorem 1 yields

$$\operatorname{vec}(\mathcal{P}_r(M+\Delta)-M) = (I_{mn} - (V_2 \otimes U_2)(V_2 \otimes U_2)^T)\operatorname{vec}(\Delta) + q(\operatorname{vec}(\Delta)) \tag{5}$$

where $q(\operatorname{vec}(\Delta)) = \operatorname{vec}(Q(\Delta))$. From the IHT update, the error matrix is

$$E^{(k)} = X^{(k)} - M = \mathcal{P}_r\Big(X^{(k-1)} - \alpha[X^{(k-1)} - M]_{\mathcal{S}}\Big) - M = \mathcal{P}_r\Big(M + E^{(k-1)} - \alpha[E^{(k-1)}]_{\mathcal{S}}\Big) - M.$$

From (5), we have

$$\begin{aligned}
e^{(k)} = \operatorname{vec}(E^{(k)}) &= (I_{mn} - WW^T)\operatorname{vec}(E^{(k-1)} - \alpha[E^{(k-1)}]_{\mathcal{S}}) + q\big(\operatorname{vec}(E^{(k-1)} - \alpha[E^{(k-1)}]_{\mathcal{S}})\big) \\
&= (I_{mn} - WW^T)(I_{mn} - \alpha S^T S)e^{(k-1)} + q\big((I_{mn} - \alpha S^T S)e^{(k-1)}\big) \\
&= (I_{mn} - WW^T)((1-\alpha)I_{mn} + \alpha S_c^T S_c)e^{(k-1)} + C_1 q(e^{(k-1)}) \\
&= \Big(I_{mn} - \alpha\Big((I_{mn} - WW^T)(I_{mn} - S_c^T S_c) + \frac{1}{\alpha}WW^T\Big)\Big)e^{(k-1)} + C_1 q(e^{(k-1)})
\end{aligned}$$

where $W = V_2 \otimes U_2 \in \mathbb{R}^{mn \times (m-r)(n-r)}$ and C_1 is some positive constant. Now, denote

$$Z_\alpha = (I_{mn} - WW^T)(I_{mn} - S_c^T S_c) + \frac{1}{\alpha}WW^T.$$

Using Lemma 10 in [2], we obtain the upper bound

$$\left\|e^{(k)}\right\|_2 \leq \big(\rho(I_{mn} - \alpha Z_\alpha) + o(1)\big)^k \left\|e^{(0)}\right\|_2$$

where $\rho(I_{mn} - \alpha Z_\alpha)$ is the spectral radius of $I_{mn} - \alpha Z_\alpha$ and is equal to the maximum magnitude of any eigenvalue of $I_{mn} - \alpha Z_\alpha$. Denote $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{mn} \geq 0$ are eigenvalues of Z_α and assume that Z_α is diagonalizable. Then, finding optimal step size α is equivalent to solving the following problem

$$\min_{\alpha} \max_{1 \leq j \leq mn} |1 - \alpha\lambda_j|. \tag{6}$$

The solution of the optimization problem (6) is given by $\alpha = \frac{2}{\lambda_1 + \lambda_{mn}}$ and the optimal rate $\rho(I - \alpha Z_\alpha) = \frac{\lambda_1 - \lambda_{mn}}{\lambda_1 + \lambda_{mn}}$. Now, using the following lemma to simplify the calculation of λ_j, we obtain $\lambda_1 = L$ and $\lambda_{mn} = \mu$.

Lemma 1. *For any $\lambda \in \Lambda(Z_\alpha)$, we have either $\lambda = \frac{1}{\alpha}$ or $\lambda = 1$ or $\lambda \in \Lambda(H)$, where $H = S_c WW^T S_c^T$.*

Proof. For any $\lambda \in \Lambda\Big((I_{mn} - WW^T)(I_{mn} - S_c^T S_c) + \frac{1}{\alpha}WW^T\Big)$, there exists $v \in \mathbb{C}^{mn}, v \neq \mathbf{0}$ such that

$$\Big((I_{mn} - WW^T)(I_{mn} - S_c^T S_c) + \frac{1}{\alpha}WW^T\Big)v = \lambda v. \tag{7}$$

Left-multiplying both sides with $(I_{mn} - WW^T)$ and recall that $W^T W = I_{(m-r)(n-r)}$, we have

$$(I_{mn} - WW^T)(I_{mn} - S_c^T S_c)v = \lambda(I_{mn} - WW^T)v.$$

Substituting back into (7), we get

$$\lambda(I_{mn} - WW^T)v + \frac{1}{\alpha}WW^T v = \lambda v \qquad \Rightarrow \qquad (\frac{1}{\alpha} - \lambda)WW^T v = \mathbf{0}.$$

Hence, we have either $\lambda = \frac{1}{\alpha}$ or $WW^T v = \mathbf{0}$. In the later case, we can substitute into (7) again to obtain

$$\lambda v = (I_{mn} - WW^T)(I_{mn} - S_c^T S_c)v = (I_{mn} - S_c^T S_c + WW^T S_c^T S_c)v. \tag{8}$$

Left-multiplying both sides with S_c and recall that $S_c S_c^T = I_{mn-s}$, we have $S_c WW^T S_c^T(S_c v) = \lambda(S_c v)$. If $S_c v = \mathbf{0}$, then plugging into (8) yields $\lambda = 1$. Otherwise, we have $\lambda \in \Lambda(S_c WW^T S_c^T)$. This completes our proof of the lemma. □

Proof of Theorem 4

The error matrix can be represented as follows

$$E^{(k+1)} = X^{(k+1)} - M = \mathcal{P}_r\Big(X^{(k)} - \alpha[X^{(k)} - M]_{\mathcal{S}}\Big) + \beta(X^{(k)} - X^{(k-1)}) - M$$
$$= \Big(\mathcal{P}_r\big(M + E^{(k)} - \alpha[E^{(k)}]_{\mathcal{S}}\big) - M\Big) + \beta\Big(E^{(k)} - (E^{(k-1)}\Big).$$

Similarly to Theorem 3, we can vectorize the above equation as

$$e^{(k+1)} = \Big(I_{mn} - \alpha Z_\alpha\Big)e^{(k)} + \beta(e^{(k)} - e^{(k-1)}) + C_1 q(e^{(k)}).$$

By stacking $e^{(k+1)}$ and $e^{(k)}$ together, the recursion can be rewritten as follows

$$\begin{bmatrix} e^{(k+1)} \\ e^{(k)} \end{bmatrix} = \underbrace{\begin{bmatrix} (1+\beta)I_{mn} - \alpha Z_\alpha & -\beta I_{mn} \\ I_{mn} & \mathbf{0} \end{bmatrix}}_{T} \begin{bmatrix} e^{(k)} \\ e^{(k-1)} \end{bmatrix} + \begin{bmatrix} C_1 q(e^{(k)}) \\ 0 \end{bmatrix}.$$

Now, using Lemma 10 in [2], we obtain the upper bound

$$\left\| \begin{bmatrix} e^{(k+1)} \\ e^{(k)} \end{bmatrix} \right\|_2 \leq \big(\rho(T) + o(1)\big)^k \left\| \begin{bmatrix} e^{(1)} \\ e^{(0)} \end{bmatrix} \right\|_2$$

where $\rho(T)$ is the spectral radius of T. Assume that Z_α is diagonalizable, then T is similar to a block diagonal matrix with 2×2 block T_j of the form [1]

$$\begin{bmatrix} 1+\beta-\alpha\lambda_j & -\beta \\ 1 & 0 \end{bmatrix}$$

for $j = 1, \ldots, mn$. Thus, the eigenvalues of T are also the eigenvalues of all blocks T_j. Finding optimal step size β is equivalent to solving the following problem

$$\min_{\alpha,\beta} \max |r| \text{ such that } r^2 - (1+\beta-\alpha\lambda_j)r + \beta = 0, \text{ for some } j \in \{1, \ldots, mn\}. \tag{9}$$

Since $\Delta = (1+\beta-\alpha\lambda)^2 - 4\beta$, it is easy to verify that

- if $\Delta \leq 0$, then $|\sigma_1| = |\sigma_2| = \sqrt{|\beta|}$,
- if $\Delta > 0$, then $\max\{|\sigma_1|, |\sigma_2|\} > \sqrt{|\beta|}$.

The optimization (9) becomes

$$\min_{\alpha,\beta} \sqrt{\beta} \qquad \text{s.t. } (1+\beta-\alpha\lambda_j)^2 - 4\beta \leq 0 \text{ for all } 1 \leq j \leq mn$$
$$\Leftrightarrow \quad \min_{\alpha,\beta} \max_j \left|1 - \sqrt{\alpha\lambda_j}\right| \qquad \text{s.t. } \beta \geq (1-\sqrt{\alpha\lambda_j})^2 \text{ for all } 1 \leq j \leq mn. \tag{10}$$

The solution of the optimization problem (10) is given by

$$\alpha = \Big(\frac{2}{\sqrt{\lambda_1} + \sqrt{\lambda_{mn}}}\Big)^2, \qquad \beta = \Big(\frac{\sqrt{\lambda_1} - \sqrt{\lambda_{mn}}}{\sqrt{\lambda_1} + \sqrt{\lambda_{mn}}}\Big)^2.$$

Finally, we obtain the optimal rate $\rho(T) = \frac{\sqrt{\lambda_1} - \sqrt{\lambda_{mn}}}{\sqrt{\lambda_1} + \sqrt{\lambda_{mn}}}$. Now, using Lemma 1, we obtain $\lambda_1 = L$ and $\lambda_{mn} = \mu$.

[1] This is shown by performing a change of basis on orthogonal space of H, following by permutations on rows and columns.

References

[1] F. Li and R.-J. Vaccaro, "Unified analysis for DOA estimation algorithms in array signal processing," *Signal Processing*, vol. 22, pp. 147–169, 1991.

[2] B. Polyak, "Some methods of speeding up the convergence of iteration methods," in *Ussr Computational Mathematics and Mathematical Physics*, 1964, vol. 4, pp. 1–17.