Supplementary Material - "On Local Linear Convergence of Projected Gradient Descent for Unit-Modulus Least Squares", Trung Vu, Raviv Raich, and Xiao Fu

SECTION A
PROOF OF LEMMA 2

Since the constraint gradients are of the form $\{\boldsymbol{e}_i \otimes \boldsymbol{S}_i(\boldsymbol{x}^*)\}_{i=1}^N$, the tangent space to $\mathcal{C}$ at $\boldsymbol{x}^*$ is given by

$$T_{\boldsymbol{x}^*}\mathcal{C} = \Big\{ \boldsymbol{y} \in \mathbb{R}^{2N} \mid \big(\sum_{i=1}^N \boldsymbol{e}_i\boldsymbol{e}_i^\top \otimes \boldsymbol{S}_i(\boldsymbol{x}^*)\big)^\top \boldsymbol{y} = \boldsymbol{0}_N \Big\}.$$

Denote $\boldsymbol{v}_i = [-x^*_{2i}, x^*_{2i-1}]^\top$ for $i = 1,\dots,N$. A basis of $T_{\boldsymbol{x}^*}\mathcal{C}$ is given by $\{\boldsymbol{e}_i \otimes \boldsymbol{v}_i\}_{i=1}^N$, i.e., the columns of $\boldsymbol{Z}$. Alternatively, $T_{\boldsymbol{x}^*}\mathcal{C}$ can be represented as

$$T_{\boldsymbol{x}^*}\mathcal{C} = \big\{ \boldsymbol{Z}\boldsymbol{u} \mid \boldsymbol{u} \in \mathbb{R}^N \big\}. \tag{50}$$

($\Rightarrow$) From Chapter 11.5 in [18], the second-order necessary condition for a stationary point $\boldsymbol{x}^*$ to be a local minimum of (10) is $\boldsymbol{y}^\top \nabla^2_{\boldsymbol{x}} L(\boldsymbol{x}^*, \boldsymbol{\gamma}) \boldsymbol{y} \geq 0$ for all $\boldsymbol{y} \in T_{\boldsymbol{x}^*}\mathcal{C}$. In other words, for any $\boldsymbol{u} \in \mathbb{R}^N$, we have

$$\begin{aligned} 0 &\leq (\boldsymbol{Z}\boldsymbol{u})^\top \big(\boldsymbol{A}^\top\boldsymbol{A} - \operatorname{diag}(\boldsymbol{\gamma}) \otimes \boldsymbol{I}_2\big)(\boldsymbol{Z}\boldsymbol{u}) \\ &= \boldsymbol{u}^\top \big(\boldsymbol{Z}^\top\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{Z} - \boldsymbol{Z}^\top(\operatorname{diag}(\boldsymbol{\gamma}) \otimes \boldsymbol{I}_2)\boldsymbol{Z}\big)\boldsymbol{u} \\ &= \boldsymbol{u}^\top \big(\boldsymbol{Z}^\top\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{Z} - \boldsymbol{Z}^\top\boldsymbol{Z}\operatorname{diag}(\boldsymbol{\gamma})\big)\boldsymbol{u} \\ &= \boldsymbol{u}^\top \big(\boldsymbol{Z}^\top\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{Z} - \operatorname{diag}(\boldsymbol{\gamma})\big)\boldsymbol{u}, \end{aligned}$$

where the second equality stems from Lemma 11 and the third equality uses the semi-orthogonality of $\boldsymbol{Z}$. Thus, we conclude that $\boldsymbol{H} \succeq \boldsymbol{0}_N$.

($\Leftarrow$) From Chapter 11.5 in [18], the second-order sufficient condition for a stationary point $\boldsymbol{x}^*$ to be a local minimum of (10) is $\boldsymbol{y}^\top \nabla^2_{\boldsymbol{x}} L(\boldsymbol{x}^*, \boldsymbol{\gamma}) \boldsymbol{y} > 0$ for all $\boldsymbol{y} \in T_{\boldsymbol{x}^*}\mathcal{C}$. By the same argument, this is equivalent to $\boldsymbol{H} \succ \boldsymbol{0}_N$.

SECTION B
PROOF OF REMARK 1

Recall that the objective function is given by $f = \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|^2/2$. By definition of the Riemannian Hessian [23], for any vector fields $U, V : \mathcal{C} \to T\mathcal{C}$ on $\mathcal{C}$, we have

$$\operatorname{Hess} f(U, V) = \langle \nabla_U \operatorname{grad} f, V \rangle, \tag{51}$$

where $\operatorname{grad} f : \mathcal{C} \to T\mathcal{C}$ is the Riemannian gradient given by

$$\operatorname{grad} f(\boldsymbol{x}) = \boldsymbol{Z}\boldsymbol{Z}^\top \nabla f(\boldsymbol{x}) = \boldsymbol{Z}\boldsymbol{Z}^\top \boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}), \tag{52}$$

for $\boldsymbol{x} \in \mathcal{C}$ and $\boldsymbol{Z}$ is the corresponding basis matrix of the tangent space to $\mathcal{C}$ at $\boldsymbol{x}$ (see Lemma 2). In addition, $\nabla_U \operatorname{grad} f$ is the covariant derivative of the vector field $\operatorname{grad} f$ in the direction of the vector field U. The covariant derivative is the orthogonal projection of the directional derivative onto the tangent space to the manifold at $\boldsymbol{x}$, i.e.,

$$\begin{aligned} \nabla_U \operatorname{grad} f(\boldsymbol{x}) &= \boldsymbol{Z}\boldsymbol{Z}^\top D_U \operatorname{grad} f(\boldsymbol{x}) \\ &= \boldsymbol{Z}\boldsymbol{Z}^\top \lim_{t \to 0} \frac{\operatorname{grad} f(\boldsymbol{x} + t\boldsymbol{u}) - \operatorname{grad} f(\boldsymbol{x})}{t}, \end{aligned} \tag{53}$$

where $\boldsymbol{u} = U(\boldsymbol{x})$. Substituting (52) into the numerator on the RHS of (53) and simplifying the expression, we obtain

$$\nabla_U \operatorname{grad} f(\boldsymbol{x}) = \boldsymbol{Z}\boldsymbol{Z}^\top \big(\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})\big),$$

where

$$\boldsymbol{B} = \sum_{i=1}^N \boldsymbol{e}_i\boldsymbol{e}_i^\top \otimes \Big(\boldsymbol{S}_i(\boldsymbol{u})\big(\boldsymbol{S}_i(\boldsymbol{x})\big)^\top + \boldsymbol{S}_i(\boldsymbol{x})\big(\boldsymbol{S}_i(\boldsymbol{u})\big)^\top \Big).$$

Now, denoting $\boldsymbol{v} = V(\boldsymbol{x})$ and evaluating (51) at $\boldsymbol{x}$ yields

$$\begin{aligned} \operatorname{Hess} f_{\boldsymbol{x}}(\boldsymbol{u}, \boldsymbol{v}) &= \boldsymbol{v}^\top \boldsymbol{Z}\boldsymbol{Z}^\top \big(\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})\big) \\ &= \boldsymbol{v}^\top \big(\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})\big), \end{aligned} \tag{54}$$

where the last equality stems from $\boldsymbol{v} \in T_{\boldsymbol{x}}\mathcal{C}$ and hence, $\boldsymbol{v} = \boldsymbol{Z}\boldsymbol{Z}^\top\boldsymbol{v}$. In the case $\boldsymbol{x} = \boldsymbol{x}^*$ is a stationary point of (10) with the Lagrange multiplier $\boldsymbol{\gamma}$, one can substituting (18) into (54) to obtain

$$\operatorname{Hess} f_{\boldsymbol{x}}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{v}^\top \big(\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{u} - \boldsymbol{B}(\operatorname{diag}(\boldsymbol{\gamma}) \otimes \boldsymbol{I}_2)\boldsymbol{x}\big). \tag{55}$$

Notice that $\boldsymbol{x} = \sum_{i=1}^N \boldsymbol{e}_i \otimes \boldsymbol{S}_i(\boldsymbol{x})$ and $\big(\boldsymbol{S}_i(\boldsymbol{u})\big)^\top \boldsymbol{S}_i(\boldsymbol{x}) = 0$ for all $i = 1,\dots,N$. Therefore, the second term on the RHS of (55) can be simplified as

$$\boldsymbol{B}(\operatorname{diag}(\boldsymbol{\gamma}) \otimes \boldsymbol{I}_2)\boldsymbol{x} = \sum_{i=1}^N \gamma_i \boldsymbol{e}_i \otimes \boldsymbol{S}_i(\boldsymbol{u}) = (\operatorname{diag}(\boldsymbol{\gamma}) \otimes \boldsymbol{I}_2)\boldsymbol{u}.$$

Substituting back into (55) and reorganizing terms, we obtain the Riemannian Hessian as

$$\operatorname{Hess} f_{\boldsymbol{x}}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{u}^\top \big(\boldsymbol{A}^\top\boldsymbol{A} - (\operatorname{diag}(\boldsymbol{\gamma}) \otimes \boldsymbol{I}_2)\big)\boldsymbol{v}. \tag{56}$$

Finally, it follows from (50) that there is a one-to-one correspondence between the tangent space $T_{\boldsymbol{x}}\mathcal{C}$ and $\mathbb{R}^N$, i.e., $\boldsymbol{u} = \boldsymbol{Z}\tilde{\boldsymbol{u}}$ and $\boldsymbol{v} = \boldsymbol{Z}\tilde{\boldsymbol{v}}$ for $\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}} \in \mathbb{R}^N$. Hence, we can define a bilinear function $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$:

$$\begin{aligned} H(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) &\triangleq \operatorname{Hess} f_{\boldsymbol{x}}(\boldsymbol{u}, \boldsymbol{v}) \\ &= (\boldsymbol{Z}\tilde{\boldsymbol{u}})^\top \big(\boldsymbol{A}^\top\boldsymbol{A} - (\operatorname{diag}(\boldsymbol{\gamma}) \otimes \boldsymbol{I}_2)\big)(\boldsymbol{Z}\tilde{\boldsymbol{v}}) \\ &= \tilde{\boldsymbol{u}}^\top (\boldsymbol{Z}^\top\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{Z} - \operatorname{diag}(\boldsymbol{\gamma}))\tilde{\boldsymbol{v}}, \end{aligned}$$

where the last equality stems from $\boldsymbol{Z}^\top\boldsymbol{Z} = \boldsymbol{I}_N$. In other words, $\operatorname{Hess} f_{\boldsymbol{x}}$ admits a compact matrix representation $\boldsymbol{H} = \boldsymbol{Z}^\top\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{Z} - \operatorname{diag}(\boldsymbol{\gamma})$.

SECTION C
PROOF OF LEMMA 3

($\Rightarrow$) Assume $\boldsymbol{x}^*$ is a fixed point of Algorithm 1 with step size $\eta > 0$, i.e.,

$$\boldsymbol{x}^* = \mathcal{P}_{\mathcal{C}}(\boldsymbol{x}^* - \eta\boldsymbol{r}), \tag{57}$$

where $\boldsymbol{r} = \boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{x}^* - \boldsymbol{b})$. We will show there exists $\boldsymbol{\gamma} \in \mathbb{R}^N$ such that for all $i = 1,\dots,N$,

$$\boldsymbol{S}_i(\boldsymbol{r}) = \gamma_i \boldsymbol{S}_i(\boldsymbol{x}^*) \tag{58}$$

and

$$\begin{cases} \gamma_i < 1/\eta & \text{if } \boldsymbol{S}_i(\boldsymbol{x}^*) \neq \boldsymbol{s}, \\ \gamma_i \leq 1/\eta & \text{if } \boldsymbol{S}_i(\boldsymbol{x}^*) = \boldsymbol{s}, \end{cases} \tag{59}$$

where we recall that $\boldsymbol{s} = [1, 0]^\top$.

For $i = 1, \ldots, N$, applying the 2-selection operator $\boldsymbol{S}_i(\cdot)$ to both side of (57) and substituting the RHS by the definition of $\mathcal{P}_{\mathcal{C}}$ in (13) yield

$$\boldsymbol{S}_i(\boldsymbol{x}^*) = \begin{cases} \frac{\boldsymbol{S}_i(\boldsymbol{x}^* - \eta \boldsymbol{r})}{\|\boldsymbol{S}_i(\boldsymbol{x}^* - \eta \boldsymbol{r})\|} & \text{if } \boldsymbol{S}_i(\boldsymbol{x}^* - \eta \boldsymbol{r}) \neq \boldsymbol{0}_2, \\ \boldsymbol{s} & \text{if } \boldsymbol{S}_i(\boldsymbol{x}^* - \eta \boldsymbol{r}) = \boldsymbol{0}_2. \end{cases} \tag{60}$$

We split (60) into two cases based on the value of $\boldsymbol{S}_i(\boldsymbol{x}^*)$. If $\boldsymbol{S}_i(\boldsymbol{x}^*) \neq \boldsymbol{s}$, then (60) implies

$$\boldsymbol{S}_i(\boldsymbol{x}^*) = \frac{\boldsymbol{S}_i(\boldsymbol{x}^* - \eta \boldsymbol{r})}{\|\boldsymbol{S}_i(\boldsymbol{x}^* - \eta \boldsymbol{r})\|} = \frac{\boldsymbol{S}_i(\boldsymbol{x}^*) - \eta \boldsymbol{S}_i(\boldsymbol{r})}{\|\boldsymbol{S}_i(\boldsymbol{x}^* - \eta \boldsymbol{r})\|},$$

which in turns can be reorganized as $\boldsymbol{S}_i(\boldsymbol{r}) = \gamma_i \boldsymbol{S}_i(\boldsymbol{x}^*)$ for

$$\gamma_i \triangleq \frac{1 - \|\boldsymbol{S}_i(\boldsymbol{x}^*) - \eta \boldsymbol{S}_i(\boldsymbol{r})\|}{\eta} < \frac{1}{\eta}. \tag{61}$$

If $\boldsymbol{S}_i(\boldsymbol{x}^*) = \boldsymbol{s}$, we consider two sub-cases:

1) If $\boldsymbol{S}_i(\boldsymbol{x}^* - \eta \boldsymbol{r}) \neq \boldsymbol{0}_2$, then by the same argument as the previous case, we obtain (61).
2) If $\boldsymbol{S}_i(\boldsymbol{x}^* - \eta \boldsymbol{r}) = \boldsymbol{0}_2$, then using the linearity of $\boldsymbol{S}_i$, we have $\boldsymbol{S}_i(\boldsymbol{r}) = \gamma_i \boldsymbol{S}_i(\boldsymbol{x}^*)$ where $\gamma_i = 1/\eta$.

In all cases, we have (58) and (59) hold. Finally, we note that the stationarity condition (18) is equivalent to $\boldsymbol{S}_i(\boldsymbol{r}) = \gamma_i \boldsymbol{S}_i(\boldsymbol{x}^*)$ for all $i = 1, \ldots, N$.

($\Leftarrow$) Assume $\boldsymbol{x}^*$ is a stationary point of (10) (i.e., (58 holds for all $i = 1, \ldots, N$) with the corresponding Lagrange multiplier $\boldsymbol{\gamma}$ satisfying (59) for all $i = 1, \ldots, N$. We will prove (57) by showing that

$$\boldsymbol{S}_i\big(\mathcal{P}_{\mathcal{C}}(\boldsymbol{x}^* - \eta \boldsymbol{r})\big) = \boldsymbol{S}_i(\boldsymbol{x}^*), \tag{62}$$

for any $i = 1, \ldots, N$.

By the definition of $\mathcal{P}_{\mathcal{C}}$ in (13), we have

$$\boldsymbol{S}_i\big(\mathcal{P}_{\mathcal{C}}(\boldsymbol{x}^* - \eta \boldsymbol{r})\big) = \begin{cases} \frac{\boldsymbol{S}_i(\boldsymbol{x}^* - \eta \boldsymbol{r})}{\|\boldsymbol{S}_i(\boldsymbol{x}^* - \eta \boldsymbol{r})\|} & \text{if } \boldsymbol{S}_i(\boldsymbol{x}^* - \eta \boldsymbol{r}) \neq \boldsymbol{0}_2, \\ \boldsymbol{s} & \text{if } \boldsymbol{S}_i(\boldsymbol{x}^* - \eta \boldsymbol{r}) = \boldsymbol{0}_2. \end{cases} \tag{63}$$

Using the linearity of $\boldsymbol{S}_i(\cdot)$ and then the stationarity condition in (58) yield

$$\begin{aligned} \boldsymbol{S}_i(\boldsymbol{x}^* - \eta \boldsymbol{r}) &= \boldsymbol{S}_i(\boldsymbol{x}^*) - \eta \boldsymbol{S}_i(\boldsymbol{r}) \\ &= \boldsymbol{S}_i(\boldsymbol{x}^*) - \eta \gamma_i \boldsymbol{S}_i(\boldsymbol{x}^*) = (1 - \eta \gamma_i) \boldsymbol{S}_i(\boldsymbol{x}^*). \end{aligned} \tag{64}$$

Since $\boldsymbol{x} \in \mathcal{C}$, $\|\boldsymbol{S}_i(\boldsymbol{x}^*)\| = 1$. Taking the norm of both sides in (64) and using (59) to remove the absolute value, we obtain

$$\begin{aligned} \|\boldsymbol{S}_i(\boldsymbol{x}^* - \eta \boldsymbol{r})\| &= \|(1 - \eta \gamma_i) \boldsymbol{S}_i(\boldsymbol{x}^*)\| \\ &= |1 - \eta \gamma| \|\boldsymbol{S}_i(\boldsymbol{x}^*)\| = 1 - \eta \gamma. \end{aligned}$$

Therefore, (63) is equivalent to

$$\boldsymbol{S}_i\big(\mathcal{P}_{\mathcal{C}}(\boldsymbol{x}^* - \eta \boldsymbol{r})\big) = \begin{cases} \boldsymbol{S}_i(\boldsymbol{x}^*) & \text{if } 1 - \eta \gamma_i \neq 0, \\ \boldsymbol{s} & \text{if } 1 - \eta \gamma_i = 0. \end{cases} \tag{65}$$

If $1 - \eta \gamma_i \neq 0$, then (62) holds trivially. If $1 - \eta \gamma_i = 0$, then $\boldsymbol{S}_i(\mathcal{P}_{\mathcal{C}}(\boldsymbol{x}^* - \eta \boldsymbol{r})) = \boldsymbol{s}$ and $\gamma_i = 1/\eta$. From (59), the latter only holds if $\boldsymbol{S}_i(\boldsymbol{x}^*) = \boldsymbol{s}$. Thus, we obtain $\boldsymbol{S}_i(\mathcal{P}_{\mathcal{C}}(\boldsymbol{x}^* - \eta \boldsymbol{r})) = \boldsymbol{S}_i(\boldsymbol{x}^*) = \boldsymbol{s}$. In both case, we have (62) holds for all $i = 1, \ldots, N$. This completes our proof of the lemma.

SECTION D
PROOF OF LEMMA 4

In this section, we show that when Conditions (C1) and (C2) in Theorem 1 hold, Condition (C3'), i.e.,

$$\eta(\lambda_1(\boldsymbol{H}) + 2\gamma_i) < 2, \tag{66}$$

for all $i = 1, \ldots, N$, is sufficient for Condition (C3). First, we prove that $\boldsymbol{D}_\eta = (\boldsymbol{I}_N - \eta \operatorname{diag}(\boldsymbol{\gamma}))^{-1}$ is PSD. Second, we show that all the eigenvalues of $\boldsymbol{D}_\eta \boldsymbol{H}$ lie between 0 and $(1 - \eta\gamma_i)^{-1}\lambda_1(\boldsymbol{H})$ (exclusively). Third, we prove that the spectral radius of $\boldsymbol{M}_\eta = \boldsymbol{I}_N - \eta \boldsymbol{D}_\eta \boldsymbol{H}$ is strictly less than 1.

In the first step, rearranging (66), we obtain $\eta\lambda_1(\boldsymbol{H})/2 < 1 - \eta\gamma_i$. By Condition (C1), we have $\lambda_1(\boldsymbol{H}) > 0$. Since $\eta > 0$, it follows that $0 < \eta\lambda_1(\boldsymbol{H})/2 < 1 - \eta\gamma_i$. Thus, the diagonal matrix $\boldsymbol{D}_\eta$ has all positive entries and hence, is a PSD matrix. In the second step, we use the inequalities for the eigenvalues of the product of two PSD matrices in [34] to obtain

$$\lambda_i(\boldsymbol{D}_\eta)\lambda_N(\boldsymbol{H}) \le \lambda_i(\boldsymbol{D}_\eta \boldsymbol{H}) \le \lambda_i(\boldsymbol{D}_\eta)\lambda_1(\boldsymbol{H}), \tag{67}$$

for all $i = 1, \ldots, N$. Since both $\boldsymbol{D}_\eta$ and $\boldsymbol{H}$ are PSD, we can lower bound the eigenvalues of $\boldsymbol{D}_\eta \boldsymbol{H}$ by $\lambda_i(\boldsymbol{D}_\eta \boldsymbol{H}) \ge \lambda_i(\boldsymbol{D}_\eta)\lambda_N(\boldsymbol{H}) > 0$. On the other hand, substituting $\lambda_i(\boldsymbol{D}_\eta) = (1 - \eta\gamma_i)^{-1}$ into the upper bound in (67) yields $\lambda_i(\boldsymbol{D}_\eta \boldsymbol{H}) \le (1 - \eta\gamma_i)^{-1}\lambda_1(\boldsymbol{H})$. Finally, using the fact that $\lambda_i(\boldsymbol{M}_\eta) = 1 - \eta\lambda_i(\boldsymbol{D}_\eta \boldsymbol{H})$ and $0 < \lambda_i(\boldsymbol{D}_\eta \boldsymbol{H}) \le (1 - \eta\gamma_i)^{-1}\lambda_1(\boldsymbol{H})$, for all $i = 1, \ldots, N$, we obtain

$$1 - \frac{\eta}{1 - \eta\gamma_i}\lambda_1(\boldsymbol{H}) \le \lambda_i(\boldsymbol{M}_\eta) < 1.$$

Now, rearranging (66) to obtain $1 - \frac{\eta}{1 - \eta\gamma_i}\lambda_1(\boldsymbol{H}) > -1$, we have all the eigenvalues of $\boldsymbol{M}_\eta$ lie between -1 and 1 (exclusively). Since the spectral radius is the maximum of the absolute values of these eigenvalues, we conclude that $\rho(\boldsymbol{M}_\eta) < 1$. This completes our proof in this section.

SECTION E
PROOF OF LEMMA 5

In the first part of this proof, we show that $\gamma_i < 1/\eta$ for all $i = 1, \ldots, N$. From Condition (C2), we have $\boldsymbol{D}_\eta = (\boldsymbol{I}_N - \eta \operatorname{diag}(\boldsymbol{\gamma}))^{-1}$ is invertible and hence, the expression of $\boldsymbol{M}_\eta$ in (24) is well-defined. In addition, from Condition (C1), $\boldsymbol{H}$ has a unique PD square root $\boldsymbol{H}^{1/2}$, with the inverse $\boldsymbol{H}^{-1/2}$. Thus, we have

$$\begin{aligned} &\boldsymbol{H}^{1/2}\boldsymbol{M}_\eta \boldsymbol{H}^{-1/2} \\ &\quad = \boldsymbol{H}^{1/2}\Big(\boldsymbol{I}_N - \eta\Big(\boldsymbol{I}_N - \eta \operatorname{diag}(\boldsymbol{\gamma})\Big)^{-1}\boldsymbol{H}\Big)\boldsymbol{H}^{-1/2} \\ &\quad = \boldsymbol{I}_N - \eta \boldsymbol{H}^{1/2}\boldsymbol{D}_\eta \boldsymbol{H}^{1/2} \triangleq \tilde{\boldsymbol{M}}_\eta. \end{aligned}$$

This shows that $\boldsymbol{M}_\eta$ and $\tilde{\boldsymbol{M}}_\eta$ are similar matrices with the same set of eigenvalues. Combining this with Condition (C3), we obtain $\rho(\boldsymbol{M}_\eta) = \rho(\tilde{\boldsymbol{M}}_\eta) < 1$. Since $\tilde{\boldsymbol{M}}_\eta$ is symmetric, it then holds that

$$\tilde{\boldsymbol{M}}_\eta = \boldsymbol{I}_N - \eta \boldsymbol{H}^{1/2}\boldsymbol{D}_\eta \boldsymbol{H}^{1/2} \prec \boldsymbol{I}_N,$$

which in turn yields $\boldsymbol{H}^{1/2}\boldsymbol{D}_\eta \boldsymbol{H}^{1/2} \succ \boldsymbol{0}_N$. By the definition of PD matrices, for any vector $\boldsymbol{u} \in \mathbb{R}^N$, it holds that $\boldsymbol{u}^\top \boldsymbol{H}^{1/2}\boldsymbol{D}_\eta \boldsymbol{H}^{1/2}\boldsymbol{u} > 0$. Alternatively, we can write

$\boldsymbol{v}^\top \boldsymbol{D}_\eta \boldsymbol{v} > 0$, where $\boldsymbol{v} = \boldsymbol{H}^{1/2}\boldsymbol{u}$. Notice that the mapping between $\boldsymbol{u}$ and $\boldsymbol{v}$ is bijective, which means $\boldsymbol{v}^\top \boldsymbol{D}_\eta \boldsymbol{v} > 0$ also holds for any $\boldsymbol{v} \in \mathbb{R}^N$. Consequently, $\boldsymbol{D}_\eta = \operatorname{diag}([(1-\eta\gamma_1)^{-1}, \dots, (1-\eta\gamma_N)^{-1}])$ must be a PD matrix. Equivalently, we have $\gamma_i < 1/\eta$ for all $i = 1, \dots, N$.

For the second part of the proof, we note that $\gamma_i < 1/\eta$, for all $i = 1, \dots, N$, are sufficient conditions for the Lagrange multiplier condition (22) in Lemma 3. Since a strict local minimum is also a stationary point of (10), $\boldsymbol{x}^*$ must be a fixed point of Algorithm 1 with the given step size η. This completes our proof of the lemma.

Section F
Proof of Lemma 6

Using the PGD update in (14) and rewriting $\boldsymbol{x}^{(k)} = \boldsymbol{x}^* + \boldsymbol{\delta}^{(k)}$, we derive a recursion on the error vector as follows

$$\begin{aligned}
\boldsymbol{\delta}^{(k+1)} &= \boldsymbol{x}^{(k+1)} - \boldsymbol{x}^* = \mathcal{P}_\mathcal{C}\Big(\boldsymbol{x}^{(k)} - \eta \boldsymbol{A}^\top\big(\boldsymbol{A}\boldsymbol{x}^{(k)} - \boldsymbol{b}\big)\Big) - \boldsymbol{x}^* \\
&= \mathcal{P}_\mathcal{C}\Big((\boldsymbol{x}^* + \boldsymbol{\delta}^{(k)}) - \eta \boldsymbol{A}^\top\big(\boldsymbol{A}(\boldsymbol{x}^* + \boldsymbol{\delta}^{(k)}) - \boldsymbol{b}\big)\Big) - \boldsymbol{x}^* \\
&= \mathcal{P}_\mathcal{C}\Big(\big(\boldsymbol{x}^* - \eta \boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{x}^* - \boldsymbol{b})\big) + (\boldsymbol{I}_{2N} - \eta \boldsymbol{A}^\top \boldsymbol{A})\boldsymbol{\delta}^{(k)}\Big) - \boldsymbol{x}^*.
\end{aligned} \tag{68}$$

Since $\boldsymbol{x}^*$ is a stationary point of (10), we have $\boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{x}^* - \boldsymbol{b}) = (\operatorname{diag}(\boldsymbol{\gamma}) \otimes \boldsymbol{I}_2)\boldsymbol{x}^*$. Then, the first term inside the projection $\mathcal{P}_\mathcal{C}$ on the RHS of (68) can be represented as

$$\begin{aligned}
\boldsymbol{x}^* - \eta \boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{x}^* - \boldsymbol{b}) &= \big(\boldsymbol{I}_{2N} - \eta \operatorname{diag}(\boldsymbol{\gamma}) \otimes \boldsymbol{I}_2\big)\boldsymbol{x}^* \\
&= \Big(\big(\boldsymbol{I}_N - \eta \operatorname{diag}(\boldsymbol{\gamma})\big) \otimes \boldsymbol{I}_2\Big)\boldsymbol{x}^* \\
&= (\boldsymbol{D}_\eta^{-1} \otimes \boldsymbol{I}_2)\boldsymbol{x}^* = (\boldsymbol{D}_\eta \otimes \boldsymbol{I}_2)^{-1}\boldsymbol{x}^*,
\end{aligned}$$

where we recall that $\boldsymbol{D}_\eta = (\boldsymbol{I}_N - \eta \operatorname{diag}(\boldsymbol{\gamma}))^{-1} \succ \boldsymbol{0}_N$ by Lemma 5. Thus, we rewrite (68) as $\boldsymbol{\delta}^{(k+1)} = \mathcal{P}_\mathcal{C}\Big((\boldsymbol{D}_\eta \otimes \boldsymbol{I}_2)^{-1}\boldsymbol{x}^* + (\boldsymbol{I}_{2N} - \eta \boldsymbol{A}^\top \boldsymbol{A})\boldsymbol{\delta}^{(k)}\Big) - \boldsymbol{x}^*$. Now let $\boldsymbol{y} = \boldsymbol{x}^* + (\boldsymbol{D}_\eta \otimes \boldsymbol{I}_2)(\boldsymbol{I}_{2N} - \eta \boldsymbol{A}^\top \boldsymbol{A})\boldsymbol{\delta}^{(k)}$ and using the modulus scale-invariance property of the projection $\mathcal{P}_\mathcal{C}((\boldsymbol{D}_\eta \otimes \boldsymbol{I}_2)^{-1}\boldsymbol{y}) = \mathcal{P}_\mathcal{C}(\boldsymbol{y})$, for $\boldsymbol{D}_\eta \succ \boldsymbol{0}_N$, we further obtain

$$\boldsymbol{\delta}^{(k+1)} = \mathcal{P}_\mathcal{C}\Big(\boldsymbol{x}^* + (\boldsymbol{D}_\eta \otimes \boldsymbol{I}_2)(\boldsymbol{I}_{2N} - \eta \boldsymbol{A}^\top \boldsymbol{A})\boldsymbol{\delta}^{(k)}\Big) - \boldsymbol{x}^*. \tag{69}$$

Finally, applying Proposition 1 with the perturbation $\boldsymbol{\delta} = (\boldsymbol{D}_\eta \otimes \boldsymbol{I}_2)(\boldsymbol{I}_{2N} - \eta \boldsymbol{A}^\top \boldsymbol{A})\boldsymbol{\delta}^{(k)}$ at $\boldsymbol{x} = \boldsymbol{x}^* \in \mathcal{C}$, we have

$$\begin{aligned}
\mathcal{P}_\mathcal{C}\Big(\boldsymbol{x}^* &+ (\boldsymbol{D}_\eta \otimes \boldsymbol{I}_2)(\boldsymbol{I}_{2N} - \eta \boldsymbol{A}^\top \boldsymbol{A})\boldsymbol{\delta}^{(k)}\Big) \\
&= \boldsymbol{x}^* + \boldsymbol{Z}\boldsymbol{Z}^\top(\boldsymbol{D}_\eta \otimes \boldsymbol{I}_2)(\boldsymbol{I}_{2N} - \eta \boldsymbol{A}^\top \boldsymbol{A})\boldsymbol{\delta}^{(k)} \\
&\quad + \boldsymbol{q}\big((\boldsymbol{D}_\eta \otimes \boldsymbol{I}_2)(\boldsymbol{I}_{2N} - \eta \boldsymbol{A}^\top \boldsymbol{A})\boldsymbol{\delta}^{(k)}\big).
\end{aligned}$$

Substituting this back into (69) yields (27). This completes the proof of the lemma.

Remark 2. *The modulus scale-invariance property of the projection leads to a more elegant analysis of asymptotic convergence rate that avoids the need of characterizing the projection at a point outside the manifold $\boldsymbol{z}_\eta^* = \boldsymbol{x}^* - \eta \boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{x}^* - \boldsymbol{b})$. This technique can also be applied to other cases, for instance, the projection onto the n-dimensional unit-sphere [11] or the Steifel manifold $\mathcal{S} = \{\boldsymbol{A} \in \mathbb{R}^{n \times r} \mid \boldsymbol{A}^\top \boldsymbol{A} = \boldsymbol{I}_r\}$.*

Section G
Proof of Lemma 7

Since $\boldsymbol{x}^{(k)}$ lies in $\mathcal{C}$, we can represent the error vector as

$$\begin{aligned}
\boldsymbol{\delta}^{(k)} = \boldsymbol{x}^{(k)} - \boldsymbol{x}^* &= \mathcal{P}_\mathcal{C}(\boldsymbol{x}^{(k)}) - \boldsymbol{x}^* \\
&= \mathcal{P}_\mathcal{C}(\boldsymbol{x}^* + \boldsymbol{\delta}^{(k)}) - \boldsymbol{x}^*.
\end{aligned} \tag{70}$$

Using Proposition 1, substituting $\mathcal{P}_\mathcal{C}(\boldsymbol{x}^* + \boldsymbol{\delta}^{(k)}) = \boldsymbol{x}^* + \boldsymbol{Z}\boldsymbol{Z}^\top\boldsymbol{\delta}^{(k)} + \boldsymbol{q}(\boldsymbol{\delta}^{(k)})$ into the RHS of (70), we obtain $\boldsymbol{\delta}^{(k)} = \boldsymbol{Z}\boldsymbol{Z}^\top\boldsymbol{\delta}^{(k)} + \boldsymbol{q}(\boldsymbol{\delta}^{(k)})$. This completes our proof of the lemma.

Section H
Auxiliary Lemmas

Lemma 11. *Given a matrix $\boldsymbol{Z} \in \mathbb{R}^{2N \times N}$ as in (19). Then for any diagonal matrix $\boldsymbol{D} \in \mathbb{R}^{N \times N}$, we have $(\boldsymbol{D} \otimes \boldsymbol{I}_2)\boldsymbol{Z} = \boldsymbol{Z}\boldsymbol{D}$.*

Proof. Recall from (39) that $\boldsymbol{Z} = \sum_{i=1}^N \boldsymbol{e}_i \boldsymbol{e}_i^\top \otimes \boldsymbol{v}_i$, where $\boldsymbol{v}_i = [-x_{2i}, x_{2i-1}]^\top$. By representing $\boldsymbol{D} = \sum_{i=1}^N D_{ii} \boldsymbol{e}_i \boldsymbol{e}_i^\top$, we have

$$\begin{aligned}
(\boldsymbol{D} \otimes \boldsymbol{I}_2)\boldsymbol{Z} &= \Big(\big(\sum_{i=1}^N D_{ii} \boldsymbol{e}_i \boldsymbol{e}_i^\top\big) \otimes \boldsymbol{I}_2\Big) \cdot \big(\sum_{j=1}^N \boldsymbol{e}_j \boldsymbol{e}_j^\top \otimes \boldsymbol{v}_j\big) \\
&= \sum_{i=1}^N \sum_{j=1}^N D_{ii}\big((\boldsymbol{e}_i^\top \boldsymbol{e}_j) \cdot \boldsymbol{e}_i \boldsymbol{e}_j^\top\big) \otimes \boldsymbol{v}_j \\
&= \sum_{i=1}^N \sum_{j=1}^N \Big(\big(\boldsymbol{e}_i \boldsymbol{e}_i^\top\big) \cdot \big(D_{jj} \boldsymbol{e}_j \boldsymbol{e}_j^\top\big)\Big) \otimes (\boldsymbol{v}_i \cdot 1) \\
&= \Big(\sum_{i=1}^N \boldsymbol{e}_i \boldsymbol{e}_i^\top \otimes \boldsymbol{v}_i\Big) \cdot \Big(\big(\sum_{i=1}^N D_{jj} \boldsymbol{e}_j \boldsymbol{e}_j^\top\big) \otimes 1\Big) \\
&= \Big(\sum_{i=1}^N \boldsymbol{e}_i \boldsymbol{e}_i^\top \otimes \boldsymbol{v}_i\Big) \cdot \big(\sum_{i=1}^N D_{jj} \boldsymbol{e}_j \boldsymbol{e}_j^\top\big) = \boldsymbol{Z}\boldsymbol{D},
\end{aligned}$$

where $\boldsymbol{e}_i^\top \boldsymbol{e}_j = 1$ if $i = j$ and $\boldsymbol{e}_i^\top \boldsymbol{e}_j = 0$ if $i \neq j$. □

Lemma 12. *For any eigenvalue λ of $\boldsymbol{Z}\boldsymbol{M}_\eta\boldsymbol{Z}^\top$, either $\lambda = 0$ or λ is an eigenvalue of $\boldsymbol{M}_\eta$. Consequently, we have*

$$\rho(\boldsymbol{Z}\boldsymbol{M}_\eta\boldsymbol{Z}^\top) = \rho(\boldsymbol{M}_\eta).$$

Proof. Let $(\lambda, \boldsymbol{u})$ be a pair of eigenvalue and eigenvector of $\boldsymbol{Z}\boldsymbol{M}_\eta\boldsymbol{Z}^\top$. Then, we have

$$\boldsymbol{Z}\boldsymbol{M}_\eta\boldsymbol{Z}^\top\boldsymbol{u} = \lambda\boldsymbol{u}. \tag{71}$$

Left-multiplying both sides of (71) by $\boldsymbol{Z}^\top$ and using the semi-orthogonality of $\boldsymbol{Z}$, we obtain $\boldsymbol{M}_\eta(\boldsymbol{Z}^\top\boldsymbol{u}) = \lambda(\boldsymbol{Z}^\top\boldsymbol{u})$. This means either $\boldsymbol{Z}^\top\boldsymbol{u} = \boldsymbol{0}_N$ or $\boldsymbol{Z}^\top\boldsymbol{u}$ is an eigenvector of $\boldsymbol{M}_\eta$. In the former case, we have $\lambda = 0$. In the latter case, we have λ is an eigenvalue of $\boldsymbol{M}_\eta$. Finally, since the spectral radius is the maximum absolute value of all eigenvalues, it is trivial that $\rho(\boldsymbol{Z}\boldsymbol{M}_\eta\boldsymbol{Z}^\top) = \rho(\boldsymbol{M}_\eta)$. □

Lemma 13. *(Rephrased from the supplemental material of [35]) Let $\{a_k\}_{k=0}^\infty \subset \mathbb{R}_+$ be the sequence defined by*

$$a_{k+1} = \rho a_k + q a_k^2 \qquad \text{for } k = 0, 1, \dots, \tag{72}$$

where $0 < \rho < 1$ *and* $q \geq 0$*. Then* $\{a_k\}_{k=0}^{\infty}$ *converges* ***monotonically*** *to* 0 *if and only if* $a_0 < \frac{1-\rho}{q}$*. A simple linear convergence bound in the case* $a_0 < \rho\frac{1-\rho}{q}$ *can be derived as*

$$a_k \leq \left(1 - \frac{a_0 q}{\rho(1-\rho)}\right)^{-1} a_0 \rho^k. \tag{73}$$

Proof. For each $k \in \mathbb{N}$, let us define $d_k = a_k/(a_0\rho^k)$. Substituting $a_k = a_0 d_k \rho^k$ into (72) and defining $\tau = a_0 q/(1-\rho)$, we obtain $d_0 = 1$ and

$$d_{k+1} = d_k + \tau(1-\rho)\rho^{k-1} d_k^2 \quad \text{for } k = 0, 1, \ldots.$$

Since $\tau(1-\rho)\rho^{k-1}d_k^2 > 0$, the sequence $\{d_k\}_{k=0}^{\infty}$ is strictly increasing and positive. Thus, using $d_{i+1} > d_i > 0$, for any $i = 0, 1, \ldots, k-1$, we have

$$\frac{1}{d_i} - \frac{1}{d_{i+1}} = \frac{d_{i+1} - d_i}{d_{i+1} d_i} < \frac{d_{i+1} - d_i}{d_i^2} = \tau(1-\rho)\rho^{i-1}.$$

Summing over $i = 0, 1, \ldots, k-1$, we obtain

$$1 - \frac{1}{d_k} < \sum_{i=0}^{k-1} \tau(1-\rho)\rho^{i-1} = \frac{\tau}{\rho}(1-\rho^k) < \frac{\tau}{\rho}. \tag{74}$$

Substituting $d_k = a_k/(a_0\rho^k)$ and $\tau = a_0 q/(1-\rho)$ into (74) and rearranging terms yield the bound on a_k in (73). □

Lemma 14. *Let* $\{b_k\}_{k=0}^{\infty} \subset \mathbb{R}_+$ *be the sequence defined by*

$$b_{k+1} \leq \rho b_k + q b_k^2 \qquad \text{for } k = 0, 1, \ldots, \tag{75}$$

where $0 < \rho < 1$ *and* $q \geq 0$*. If* $b_0 < \frac{1-\rho}{q}$*, then* $\{b_k\}_{k=0}^{\infty}$ *converges to* 0*. If* $b_0 < c \triangleq \rho\frac{1-\rho}{q}$*, then for any integer* $k \geq 0$*, we have*

$$b_k \leq \left(1 - \frac{b_0}{c}\right)^{-1} b_0 \rho^k.$$

Proof. Let us define a surrogate sequence $\{a_k\}_{k=0}^{\infty}$ that upper-bounds $\{b_k\}_{k=0}^{\infty}$ as: $a_0 = b_0$ and $a_{k+1} = \rho a_k + q a_k^2$. First, we prove by induction that

$$b_k \leq a_k, \quad \forall k \in \mathbb{N}. \tag{76}$$

The base case when $k = 0$ holds trivially as $b_0 = a_0$. In the induction step, given $b_k \leq a_k$ for an integer $k \geq 0$, we have

$$b_{k+1} \leq \rho b_k + q b_k^2 \leq \rho a_k + a_k^2 = a_{k+1}.$$

By the principle of induction, (76) holds for all $k \in \mathbb{N}$. Now, by Lemma 13, we have $b_k \leq a_k \leq \big(1 - \frac{a_0 q}{\rho(1-\rho)}\big)^{-1} a_0 \rho^k = \big(1 - \frac{b_0 q}{\rho(1-\rho)}\big)^{-1} b_0 \rho^k$. This completes our proof of the lemma. □

Section I
Region of Convergence

This subsection demonstrates the region of local convergence for PGD in a 2-D setting. Since $N = 1$ in this case, the constraint set $\mathcal{C}$ is indeed a 2-D circle. As can be seen from Fig. 7, the least squares objective has an unconstrained global minimum at $\boldsymbol{x}^*_{unc} = [0.7, 0.2]^\top$, with $\boldsymbol{A} = \text{diag}([5, 1])$ and $\boldsymbol{b} = [3.5, 0.2]^\top$. Using Lemma 1, we can find the four stationary points of the 2-D UMLS problem by solving the following system of non-linear equations

$$\begin{cases} x_1^2 + x_2^2 = 1, \\ 25x_1 - 17.5 = \gamma x_1, \\ x_2 - 0.2 = \gamma x_2. \end{cases}$$

Moreover, based on the positivity of the reduced Riemannian Hessian $h = 25x_2^2 + x_1^2 - \gamma$ (which is a scalar in the 2-D setting), one can apply Lemma 2 to determine the two local maxima (purple hexagrams) and two local minima (green asterisk and red diamond). Additionally, for each local minima, the rate of convergence is given by $\rho_\eta = 1 - \eta h/(1 - \eta\gamma)$, with the maximum possible step size $\eta_{\max} = 2/(h + 2\gamma)$. In Fig. 7, we pick $\eta = 0.0755$ and compute the theoretical region of convergence for each local minima using (30). On the other hand, the empirical region of convergence is obtained follows. First, we run PGD with $\eta = 0.0755$ and 1000 different initialization uniformly distributed on the unit circle. Next, we check whether the algorithm stops inside the theoretical region of convergence after 1000 iterations to determine if it converges to the corresponding local minimum. Finally, we color the initialization points by the color of the corresponding local minimum PGD converges to (either green or red). While Fig. 7 verifies that our theoretical region of convergence falls inside the empirical region of convergence, it also reveals that our bound is conservative in this example.

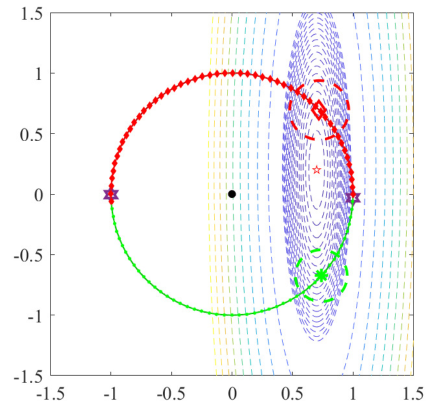

Fig. 7: A 2-D illustration of the region of convergence given by the constant $c_0(\boldsymbol{x}^*, \eta)$ in (30). On the circle, the two purple hexagrams denote the local maxima, while the green asterisk and the red diamond denote the local minima of the problem. The red star located inside the circle is the solution to the unconstrained least squares. For a given fixed step size η, each local minimum is associated with (i) an estimated region of convergence (dashed circle) given by $c_0(\boldsymbol{x}^*, \eta)$ and (ii) an empirical region of convergence (circular arc with matching color) given by running PGD with the fixed step size η and initialization at a given point on the circle to verify which local minimum it converges to.